

DEGREE OF SYMMETRY OF A HOMOTOPY REAL PROJECTIVE SPACE

BY

H. T. KU, L. N. MANN, J. L. SICKS AND J. C. SU⁽¹⁾

Abstract. The degree of symmetry $N(M)$ of a compact connected differentiable manifold M is the maximum of the dimensions of the compact Lie groups which can act differentiably and effectively on it. It is well known that $N(M) \leq \dim SO(m+1)$, for an m -dimensional manifold, and that equality holds only for the standard m -sphere and the standard real projective m -space. W. Y. Hsiang has shown that for a high dimensional exotic m -sphere M , $N(M) < m^2/8 + 1 < (\frac{1}{2}) \dim SO(m+1)$, and that $N(M) = m^2/8 + 7/8$ for some exotic m -spheres. It is shown here that the same results are true for exotic real projective spaces.

0. Introduction. The *degree of symmetry* $N(M)$ of a compact connected differentiable m -manifold M^m is the maximum of the dimensions of the compact Lie groups which can act effectively and differentiably on M . It is well known that

$$N(M^m) \leq m(m+1)/2,$$

and that if $N(M^m) = m(m+1)/2$, then M is diffeomorphic to the standard sphere S^m or the standard real projective space RP^m [3]. In [6] W. Y. Hsiang showed that if Σ^m is an exotic sphere ($m \geq 40$), then

$$N(\Sigma^m) < m^2/8 + 1.$$

Hsiang's result is best possible in the sense that

$$N(\Sigma_0^{8k+1}) = m^2/8 + 7/8$$

where Σ_0^{8k+1} is the Kervaire sphere of dimension $m = 8k + 1$.

In this paper we complement Hsiang's result by showing that if M is a homotopy real projective m -space ($m \geq 72$), not diffeomorphic to RP^m , then

$$N(M) < m^2/8 + 1.$$

Again this bound is best possible, as we exhibit an exotic homotopy real projective m -space (whose universal covering space is actually Σ_0^{8k+1} , $m = 8k + 1$) with degree of symmetry $m^2/8 + 7/8$.

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Let us comment briefly on our approach. Suppose M is a homotopy RP^m with $N(M) \geq m^2/8 + 1$ and let G be a compact connected Lie group of dimension $N(M)$ acting effectively and differentiably on M . Then the action of G on M can be lifted to an effective and differentiable action of a covering group $\tilde{G}$ of G on the universal covering space $\tilde{M}$ of M [9]. Now $\tilde{M}$ is a homotopy m -sphere and

$$N(\tilde{M}) \geq \dim \tilde{G} = \dim G = N(M) \geq m^2/8 + 1.$$

Hence by the previously mentioned result of W. Y. Hsiang, $\tilde{M}$ must be a standard sphere. However, although M is covered by the standard sphere S^m the deck involution on S^m could conceivably be exotic, and hence M might be an exotic homotopy RP^m . Our approach, then, is to establish a structure theorem directly for actions of compact Lie groups on homotopy RP^m 's. We show that if M^m is a homotopy RP^m admitting an effective differentiable action of a compact connected Lie group with S^k ($k \neq 1, 3$) as principal orbit, then M is diffeomorphic to RP^m . The proof of the structure theorem seems to hinge on the fact that the Whitehead group of the group Z_2 is trivial. Our structure theorem is analogous to one for homotopy spheres used by W. Y. Hsiang in [6] and with it we are able to establish our bound for the degree of symmetry of exotic RP^m 's.

For the sake of clarity we collect some preliminaries in the next two sections.

1. Preliminaries. Let G be a compact connected Lie group acting smoothly on a compact connected manifold M such that the fixed-point set $F(G, M)$ is non-empty. Choose a base point $x_0 \in F(G, M)$, let $\tilde{M}$ be the universal covering space of M represented as homotopy classes of paths in M starting at x_0 . Then there is a natural smooth action of G on $\tilde{M}$ so that the projection map $p: \tilde{M} \rightarrow M$ is equivariant. It follows that for $y \in \tilde{M}$ and $x = p(y) \in M$, we always have $G_y \subset G_x$. But we can say more. In fact, G_y is normal in G_x and G_x/G_y is isomorphic to a subgroup of the fundamental group $\pi_1 = \pi_1(M, x_0)$ of M . For if $g \in G_x$, then $p(gy) = gp(y) = gx = x = p(y)$. Hence there is a unique $\alpha_g \in \pi_1$ such that $gy = y\alpha_g$. (We write the action of π_1 on $\tilde{M}$ on the right.) Then $\theta: g \mapsto \alpha_g$ is a homomorphism of G_x into π_1 with $\text{Ker } \theta = G_y$. In particular, as G_x is compact and π_1 discrete, G_x/G_y is finite. We can decide when $G_x = G_y$ as follows. Since G is assumed to be connected, it is easily seen that the lifted action of G on $\tilde{M}$ commutes with the action of π_1 . Hence π_1 acts on the orbit space $\tilde{M}/G$ and the orbit space of this action can be identified with M/G . Now let $y \in \tilde{M}$ and $[y] \in \tilde{M}/G$ its orbit. It is easily seen that

$$(\pi_1)_{[y]} = \theta(G_x/G_y).$$

Therefore $G_y = G_x$ iff $[y] \in \tilde{M}/G$ is free under the action of π_1 , and this would always be the case if G_x is connected. We will be interested in applying this in two extreme cases:

(1.1) *If the principal isotropy subgroup type (H) of (G, M) is connected, then the action $(G, \tilde{M})$ also has (H) as principal isotropy subgroup type.*

(1.2) *Since G is connected, $F(G, \tilde{M}) = p^{-1}(F(G, M))$.*

2. More preliminaries. Let (Z_2, X) be a smooth involution where X is a contractible n -manifold with boundary ∂X such that $(Z_2, \partial X)$ is free. Then

(2.1) *The fixed-point set $F(Z_2, X)$ is precisely a single point in the interior of X .*

(2.2) *The orbit space X/Z_2 is contractible.*

(2.3) *Let $D^n \subset \text{Int } X$ be a disk around the fixed-point. Then the inclusion*

$$RP^{n-1} = \partial D^n / Z_2 \rightarrow X - \text{Int } D^n / Z_2$$

is a homotopy equivalence.

(2.4) $\partial X / Z_2$ is a manifold having the same integral cohomology as RP^{n-1} . Moreover, if ∂X is simply connected and $n \geq 6$, then the action $(Z_2, \partial X)$ is equivalent to the standard antipodal involution (Z_2, S^{n-1}) , and hence $\partial X / Z_2$ is diffeomorphic to RP^{n-1} .

Proof. By P. A. Smith theory $F = F(Z_2, X)$ is acyclic over Z_2 . Since $(Z_2, \partial X)$ is free, F is a manifold without boundary. It follows that F can only be a single point, say $x_0 \in \text{Int } X$. Let $D^n \subset \text{Int } X$ be a disk around it. Consider the inclusion $\partial D^n \rightarrow X - \text{Int } D^n$. From the exact triangle

$$\begin{array}{ccc} H_*(\partial D^n) & \longrightarrow & H_*(X - \text{Int } D^n) \\ & \swarrow \quad \searrow & \\ & H_*(X - \text{Int } D^n, \partial D^n) & \\ & \parallel & \\ & H_*(X, D^n) = 0 & \end{array}$$

we see that $H_*(\partial D^n) \rightarrow H_*(X - \text{Int } D^n)$ is an isomorphism. Also $X - \text{Int } D^n$ is simply connected by the Van Kampen theorem. It follows (e.g. by a spectral sequence argument) that both

$$H_*(\partial D^n / Z_2) \rightarrow H_*(X - \text{Int } D^n / Z_2)$$

and

$$\pi_1(\partial D^n / Z_2) \rightarrow \pi_1(X - \text{Int } D^n / Z_2)$$

are isomorphisms. Therefore by Whitehead's theorem,

$$RP^{n-1} = \partial D^n / Z_2 \rightarrow X - \text{Int } D^n / Z_2$$

is a homotopy equivalence.

Now consider the other end ∂X of $X - \text{Int } D^n$. We have the exact triangle

$$\begin{array}{ccc} H_*(\partial X) & \longrightarrow & H_*(X - \text{Int } D^n) \\ & \swarrow \quad \searrow & \\ & H_*(X - \text{Int } D^n, \partial X) & \end{array}$$

But Poincaré duality gives

$$H_*(X - \text{Int } D^n, \partial X) = H^*(X - \text{Int } D^n, \partial D^n) = H^*(X, D^n) = 0.$$

Thus again $H_*(\partial X) \rightarrow H_*(X - \text{Int } D^n)$ and hence $H_*(\partial X/Z_2) \rightarrow H_*(X - \text{Int } D^n/Z_2)$ are isomorphisms. Thus $H_*(\partial X/Z_2) = H_*(RP^{n-1})$. If ∂X is simply connected, then

$$\pi_1(\partial X/Z_2) \rightarrow \pi_1(X - \text{Int } D^n/Z_2)$$

would also be an isomorphism and hence the inclusion $\partial X/Z_2 \rightarrow X - \text{Int } D^n/Z_2$ is also a homotopy equivalence. But this means $X - \text{Int } D^n/Z_2$ is an h -cobordism between $\partial X/Z_2$ and RP^{n-1} . As $\text{Wh}(Z_2) = 0$, we conclude by the s -cobordism theorem that $X - \text{Int } D^n/Z_2$ is diffeomorphic to $RP^{n-1} \times I$ when $n \geq 6$. This completes the proof of (2.1) through (2.4).

3. The Structure Theorem. Let (G, M) be a smooth action of a compact connected Lie group on a compact connected manifold. Assume that all the orbits of the action are of uniform dimension. Then the connectedness of M implies that all the identity components G_x^0 of the isotropy subgroups G_x are conjugate, say of the same type (H) . Let $P = F(H, M)$ be the fixed-point set of H , $N = N(H, G)$ the normalizer of H in G and $K = N/H$ the quotient group. Then P is a submanifold of M invariant under N , so we have a natural action of K on P . On the other hand, there is the natural right translation of K on the homogeneous space G/H . Therefore we can form the space $G/H \times_K P$. This is a smooth manifold (since K acts freely on G/H) on which G acts naturally via left translations on G/H . It is easily seen that

$$G/H \times_K P \rightarrow M, \quad [gH, x] \mapsto gx$$

is a diffeomorphism identifying M with $G/H \times_K P$ as G -manifolds. With this representation of M , for a point $m = gx$ with $g \in G$, $x \in P$, let K_x be the isotropy subgroup of x in the action (K, P) , $\tilde{K}_x \subset N$ the pull-back of K_x under the projection $N \rightarrow K$. Then

$$G_{gx} = g\tilde{K}_xg^{-1}.$$

Thus G_{gx} is of type H iff $x \in P$ is free under K . For example if all orbits of G on M are actually of the same type H , then K acts freely on P and we get M fibered over P/K with fiber G/H and group K , an observation due to A. Borel. The above situation, where each K_x is permitted to be a finite group, is discussed in [2].

We shall need a slightly more general case than the above. Namely we assume G has fixed-point set $F = F(G, M)$ but all other orbits are of the same dimension. Let (H) be the principal orbit type. We assume that $G/H = S^k$ is a sphere of dimension $k \neq 1, 3$. Let U be an equivariant tubular neighborhood of F . Then G acts on $M - \text{Int } U$ with uniform dimensional orbits and we can apply the above to represent $M - \text{Int } U$ by $G/H \times_K P$, where $P = F(H, M - \text{Int } U)$. Now we put U back in. We have $U \rightarrow F$ a D^p bundle over F with group G acting on D^p via normal repre-

sentation, where p is the codimension of F in M . Thus (G, D^p) is a linear action with principal orbit type $G/H = S^k$. According to [4, Lemma 1, p. 425], if $k \neq 1, 3$, then $p = k + 1$ and the linear action of G on the unit sphere $S^{p-1} = S^k$ is just the natural translation of G on $G/H = S^k$. Now on ∂U , the diagram

$$\begin{array}{ccc} G/H \times_K \partial P & = & S^k \times_K \partial P \longrightarrow \partial U \\ \downarrow & & \downarrow \\ \partial P/K & \longrightarrow & F \end{array}$$

shows that the sphere bundle $\partial U \rightarrow F$ has $\partial P \rightarrow \partial P/K$ as its associated principal bundle. It follows that the disk bundle $U \rightarrow F$ can be identified with

$$D^{k+1} \times_K \partial P \rightarrow \partial P/K,$$

and we may represent M as

$$M = S^k \times_K P \cup D^{k+1} \times_K \partial P,$$

identified along $S^k \times_K \partial P$. Notice that from this representation, the orbit space can be given by

$$M/G = P/K \cup [0, 1] \times \partial P/K,$$

with $(1) \times \partial P/K \subset [0, 1] \times \partial P/K$ attached to $\partial P/K \subset P/K$. (For more details, see [8].)

We are ready to prove the structure theorem.

STRUCTURE THEOREM. *Let G be a compact connected Lie group acting smoothly on a homotopy real projective m -space M . Suppose the principal orbit of the action is a k -sphere S^k with $k \neq 1, 3$. Then*

(i) *There are precisely three types of orbits. Namely the sphere S^k , the real projective k -space RP^k , and fixed-points. Moreover there is exactly one orbit of type RP^k .*

(ii) *The fixed-point set $F = F(G, M)$ has the integral homology of RP^n , where $n = m - k - 1$. If $\pi_1(F) = Z_2$ and $n \geq 5$, then F is actually diffeomorphic to RP^n .*

(iii) *Let $Y = M/G$ be the orbit space of the action (G, M) , $y_0 \in M/G$ the orbit of type RP^k (see (i)) and V a neighborhood of y_0 in M/G . Then Y is a contractible space and (Y, y_0) a relative manifold of dimension $n + 1 = m - k$. More precisely, $Y - \text{Int } V$ is a smooth manifold with boundary $\partial(Y - \text{Int } V) = F \cup RP^n$, the inclusion $RP^n \rightarrow Y - \text{Int } V$ is a homotopy equivalence and V is a cone over RP^n with y_0 as vertex.*

(iv) *If $m \geq 5$, then M is diffeomorphic to RP^m .*

Proof. (i), (ii), and (iii) are relatively simple. First of all, there must be a fixed-point. For if not, then since M has no rational cohomology, Theorem 4 of [4] applies to assert that the orbits of the action have uniform dimension. As mentioned above, we can write M as

$$M = G/H \times_K P = S^k \times_K P.$$

Consider the homotopy exact sequence

$$\cdots \rightarrow \pi_{k+1}(M) \rightarrow \pi_k(K) \rightarrow \pi_k(S^k \times P) \rightarrow \pi_k(M) \rightarrow \cdots.$$

Since K acts freely on S^k , it has rank 1 and so $\pi_k(K)$ is finite, for $k \neq 1, 3$ (the identity component K^0 is either S^1 or S^3). As $\pi_k(S^k \times P)$ is infinite, we have $\pi_k(M) \neq 0$. This is impossible since $k < m$ and M is a homotopy RP^m .

Thus we can apply the results of §1. Let $\tilde{M}$ be the universal covering of M and $(G, \tilde{M})$ the lifted action. Since the principal orbit $G/H = S^k$ is simply connected, H is connected, and (1.1) implies that $(G, \tilde{M})$ also has principal orbit S^k . Of course $\tilde{M}$ is a homotopy sphere and actions of this kind are completely understood by the work of W. C. Hsiang and W. Y. Hsiang [4]. For example there are only orbits of type (H) and fixed-points. Thus the only possible orbit types of (G, M) are (H) , (G) and (G_x) with $G_x/H = \pi_1(M) = Z_2$, i.e. the only possible orbits are spheres S^k , fixed-points, and projective spaces RP^k .

Let $X = \tilde{M}/G$ be the orbit space of $(G, \tilde{M})$. We know that $\pi_1(M) = Z_2$ acts on X with orbit space $X/Z_2 = M/G = Y$, and a point $[x] = y_0 \in Y$ has orbit RP^k iff $x \in X$ is a fixed-point of Z_2 . Now by [4], X is a contractible manifold of dimension $m - k = n + 1$ with $\partial X = F(G, \tilde{M})$. By (1.2), $F(G, \tilde{M}) = p^{-1}(F(G, M))$, hence Z_2 acts freely on ∂X . We can then apply (2.1) to conclude that $F(Z_2, X)$ is precisely a single point $x_0 \in X$. Hence there is exactly one orbit of type RP^k .

Let D^{n+1} be a disk around x_0 . Then $V = D^{n+1}/Z_2 \subset Y$ is a neighborhood of y_0 in Y and V is a cone over $RP^n = \partial D^{n+1}/Z_2$ with y_0 as vertex. By (2.2), Y is a contractible space and $Y - \text{Int } V$ is a manifold with boundary $\partial(Y - \text{Int } V) = \partial X/Z_2 \cup RP^n = F \cup RP^n$. Statements (ii) and (iii) now follow from (2.2) through (2.4).

Now we proceed to prove (iv), which is our main concern. Let U be an equivariant tubular neighborhood of F . As we have seen above, we have

$$M - \text{Int } U = S^k \times_K P, \quad U = D^{k+1} \times_K \partial P,$$

and

$$M = S^k \times_K P \cup D^{k+1} \times_K \partial P.$$

We know the orbit of type RP^k is determined by a point $x_0 \in P$ such that $K_{x_0} = Z_2$, and this point projects, of course, to the point $y_0 \in Y$ in the orbit space. Now by the slice theorem, the orbit of x_0 in P has a tubular neighborhood W in P given by $W = K \times_{Z_2} D^{n+1}$. The singular orbit RP^k has therefore a tubular neighborhood $\hat{W}$ in M given by

$$\hat{W} = S^k \times_K (K \times_{Z_2} D^{n+1}) = S^k \times_{Z_2} D^{n+1}$$

and $\hat{W}$ projects to $\hat{W}/G = D^{n+1}/Z_2 = V$, the cone neighborhood of $y_0 \in Y$. Hence $Y - \text{Int } V$ is covered by

$$M - \text{Int } \hat{W} = S^k \times_K (P - \text{Int } W) \cup D^{k+1} \times_K \partial P.$$

Now consider the principal K -bundle

$$\xi: P - \text{Int } W \rightarrow (P - \text{Int } W/K) \sim (Y - \text{Int } V).$$

Here “ $\sim$ ” means of the same homotopy type, as $Y - \text{Int } V$ is obtained from $P - \text{Int } W/K$ by attaching a collar $[0, 1] \times \partial P/K$. Now $\partial V = RP^n \subset Y - \text{Int } V$ is a homotopy equivalence by (iii). Over ∂V , the bundle ξ is given by

$$\xi|_{\partial V}: \partial W = K \times_{Z_2} S^n \rightarrow S^n/Z_2 = RP^n = \partial V.$$

This means the bundle $\xi|_{\partial V}$ has a Z_2 -reduction. Since $\partial V \rightarrow Y - \text{Int } V$ is a homotopy equivalence, there is a unique Z_2 -reduction of ξ which extends the reduction of $\xi|_{\partial V}$. That is, there is a principal Z_2 -bundle $Q \rightarrow Y - \text{Int } V$, with Q a smooth manifold with boundary $\partial Q = S^n \cup B$, over $\partial V = RP^n$ and $F \subset Y - \text{Int } V$ respectively, and an identification $K \times_{Z_2} Q = P - \text{Int } W$ compatible with the identification $K \times_{Z_2} S^n = \partial W$. We can therefore represent P as

$$P = K \times_{Z_2} Q \cup K \times_{Z_2} D^{n+1} = K \times_{Z_2} R,$$

where $R = Q \cup D^{n+1}$ is obtained from Q by adjoining a disk along S^n . Notice R has a smooth action of Z_2 with exactly a single fixed-point. Now we can write M as

$$\begin{aligned} M &= S^k \times_K P \cup D^{k+1} \times_K \partial P \\ &= S^k \times_K (K \times_{Z_2} R) \cup D^{k+1} \times_K (K \times_{Z_2} \partial R) \\ &= S^k \times_{Z_2} R \cup D^{k+1} \times_{Z_2} \partial R = \partial(D^{k+1} \times R)/Z_2. \end{aligned}$$

The diagram

$$\begin{array}{ccc} S^n & \longrightarrow & Q \\ \downarrow & & \downarrow \\ RP^n & \longrightarrow & Y - \text{Int } V \end{array}$$

and the fact that $RP^n \subset Y - \text{Int } V$ is a homotopy equivalence implies that $S^n \subset Q$ is a homotopy equivalence. Hence $R = Q \cup D^{n+1}$ is contractible and so is $D^{k+1} \times R$. Thus we have a smooth action of Z_2 on a contractible manifold, which is free on the boundary. Since $\partial(D^{k+1} \times R)$ is necessarily simply connected (e.g. by the Van Kampen theorem), we conclude from (2.4) that M is diffeomorphic to RP^m when $m \geq 5$. This completes the proof of the Structure Theorem.

4. Main results. With the Structure Theorem we are now able to prove the following theorem:

THEOREM. *If M^m is an exotic HRP^m ($m \geq 72$), then*

$$N(M) < m^2/8 + 1.$$

Proof. Suppose M^m is a HRP^m with $N(M) \geq m^2/8 + 1$. We proceed to show that M^m is standard. Let G be a compact connected Lie group of dimension $\geq m^2/8 + 1$ acting effectively and differentiably on M ; let H be a principal isotropy subgroup of the action. Let Σ^m be the universal covering space of M^m . The action of G on M lifts to an effective differentiable action of a covering group $\tilde{G}$ of G on Σ^m . Now $\dim \tilde{G} = \dim G$.

Case A. $\dim G/H = m$.

In this case the action of G on M , and hence, $\tilde{G}$ on Σ^m is transitive. However, all transitive effective actions of compact connected Lie groups on spheres have been classified [11], [1], [13]. In all cases Σ^m is diffeomorphic to S^m . We obtain the principal Z_2 -bundle

$$\begin{array}{c} S^m = \tilde{G}_1/\tilde{H}_1 \\ \downarrow \tilde{H}_0/\tilde{H}_1 \\ M^m = \tilde{G}_1/\tilde{H}_0 \end{array}$$

where $\tilde{G}_1 = SO(m+1)$, $SU((m+1)/2)$ or $Sp((m+1)/4)$ and $\tilde{H}_1 = SO(m)$, $SU((m-1)/2)$ or $Sp((m-3)/4)$ (respectively) standardly imbedded. Now the free action of $\tilde{H}_0/\tilde{H}_1$ on S^m is the restriction of the orthogonal action of $N(\tilde{H}_1, \tilde{G}_1)/\tilde{H}_1$ on S^m . It follows that M^m is diffeomorphic to RP^m .

Case B. $\dim G/H = m-1$.

Now $\tilde{G}$ acts on Σ^m with principal orbit also of codimension one. H. C. Wang has classified the groups acting effectively on spheres with principal orbit of codimension one [14], [6, p. 355]. It follows that $\tilde{G}$ is locally isomorphic to a subgroup of one of the following:

- (i) $SO(t) \times SO(2)$, $m = 2t - 1$,
- (ii) $U(t) \times U(2)$, $m = 4t - 1$,
- (iii) $Sp(t) \times Sp(2)$, $m = 8t - 1$.

However in all cases,

$$\dim G = \dim \tilde{G} < m^2/8 + 1,$$

which is a contradiction.

Case C. $\dim G/H \leq m-2$.

Now

$$\dim G \geq m^2/8 + 1 \geq ((m^2 + 8)/8(m-2)) \dim G/H.$$

Hence

$$(4.1) \quad \dim G \geq r \dim G/H$$

where

$$(4.2) \quad r = (m^2 + 8)/8(m-2).$$

It follows from results of W. Y. Hsiang [7, Propositions 1 and 2] that there exists a simple connected normal subgroup G_1 of G such that

$$(4.3) \quad \dim G_1 \geq r[m(G_1)],$$

and

$$(4.4) \quad \dim G_1 + \dim N(H_1, G_1)/H_1 \geq r \dim G_1/H_1,$$

where

- (i) $m(G_1)$ = smallest positive codimension of proper subgroups of G_1 ,
- (ii) $H_1 = G_1 \cap H$,
- (iii) $N(H_1, G_1)$ = normalizer of H_1 in G_1 .

Since m , and consequently r , are large it follows from (4.3) and a knowledge of the maximal dimensions of proper subgroups of the exceptional Lie groups [8] that G_1 is a classical Lie group. Therefore G_1 is locally isomorphic to $SO(n)$, $SU(n)$, or $Sp(n)$. We consider only the case where G_1 is locally isomorphic to $SO(n)$ as the other two cases are similar and less difficult.

Since $m(SO(n)) = n - 1$, it follows from (4.3) that

$$n(n-1)/2 = \dim SO(n) \geq ((m^2+8)/8(m-2))(n-1).$$

and

$$(4.5) \quad n > m/4$$

after simplifying. Since we are assuming that $m \geq 72$ it follows from (4.5) that

$$(4.6) \quad (n-1)^2 \geq 4m.$$

Now M has zero first rational Pontrjagin class and with (4.6) we may conclude that H_1^0 , the identity component of H_1 , is locally isomorphic to $SO(n-k)$, $1 \leq k \leq n$ [5, Theorem 2.1]. We show $k \leq 4$. By the underlying hypothesis of Case C, $\dim G/H \leq m-2$. Hence

$$m-2 \geq \dim G_1/H_1 = \dim SO(n) - \dim SO(n-k)$$

and consequently

$$(4.7) \quad m \geq nk - k(k+1)/2 + 2.$$

Combining (4.5) with (4.7) yields

$$(4.8) \quad 4n > nk - k(k+1)/2 + 2$$

and since n is large it follows that $k \leq 4$. For $k=3, 4$ we may employ (4.4) directly to obtain an easy contradiction. Consider then $k=2$. From (4.4) we obtain

$$\dim SO(n) + \dim SO(2) \geq ((m^2+8)/8(m-2)) \dim G_1/H_1$$

or

$$(4.9) \quad n(n-1)/2 + 1 \geq ((m^2+8)/8(m-2))(2n-3).$$

Now it follows from (4.5) and (4.7) that $m/4 < n \leq (m+1)/2$. Let

$$f(n) = n(n-1)/2 + 1 - r(2n-3)$$

where, as we recall, $r = (m^2+8)/8(m-2)$. We proceed to reach a contradiction to (4.9) by showing that $f(n) < 0$ for $m/4 < n \leq (m+1)/2$.

It is easily checked that $f((m+1)/2) < 0$ and $f(m/4) < 0$. Since $f(n)$ is concave upward, the contention follows.

We are left with $k=1$. The almost effective action of $SO(n)$ on M may be lifted to an almost effective action of $SO(n)$ on $\tilde{M}$, the universal covering space of M . Since $\tilde{M}$ is a homotopy sphere and since $(n-1)^2 \geq 4m$, it follows from a result of the Hsiang brothers [5, Theorem 3.1] that $SO(n)$ acts on $\tilde{M}$ with principal orbit S^{n-1} and with a nonempty fixed-point set. From our considerations in the last section we know now that $SO(n)$ acts on M with a fixed-point. Now the principal orbit of the action of $SO(n)$ on M is either S^{n-1} or RP^{n-1} . However, since $SO(n)$ acts linearly in a neighborhood of a fixed-point of M , it follows that the principal orbit is actually S^{n-1} . Finally the Structure Theorem applies and M is diffeomorphic to RP^m .

EXAMPLE. We employ the Hirzebruch-Brieskorn construction of homotopy spheres to show that this theorem is best possible. Let $\tilde{M}^{8k+1}$ be the intersection of the complex hypersurfaces

$$Z_1^2 + Z_2^2 + \cdots + Z_{4k+1}^2 + Z_{4k+2}^2 = 0$$

and

$$|Z_1|^2 + |Z_2|^2 + \cdots + |Z_{4k+2}|^2 = 1$$

in C^{4k+2} . It is known that $\tilde{M}^{8k+1}$ is diffeomorphic to the Kervaire sphere Σ_0^{8k+1} (see, for example, [12, pp. 55–56]). It is easily seen that a standard subgroup $O(4k+1)$ in $U(4k+2)$ leaves $\tilde{M}^{8k+1}$ invariant in C^{4k+2} . In addition there is an S^1 action on $\tilde{M}^{8k+1}$ which commutes with the $O(4k+1)$ action. Namely, if $w = e^{i\theta} \in S^1$,

$$w(Z_1, Z_2, \dots, Z_{4k+1}, Z_{4k+2}) = (e^{3i\theta}Z_1, e^{3i\theta}Z_2, \dots, e^{3i\theta}Z_{4k+1}, e^{2i\theta}Z_{4k+2}).$$

We obtain an almost effective action of $O(4k+1) \times S^1$ on $\tilde{M}$. Consider the involution $t = e^{i\pi} = -1$ in S^1 . Clearly t acts freely on $\tilde{M}$. Let A be the group of order two generated by t and let $M^{8k+1} = \tilde{M}^{8k+1}/A$. Now M is a homotopy RP^{8k+1} and, since its universal covering space $\tilde{M}$ is an exotic sphere, M is not diffeomorphic to RP^{8k+1} . On the other hand, the almost effective action of $O(4k+1) \times S^1$ on $\tilde{M}$ can be pushed down to an almost effective action on M . Hence if $m = 8k+1$,

$$N(M) \geq \dim [O(4k+1) \times S^1] = 8k^2 + 2k + 1 = m^2/8 + 7/8.$$

REMARKS. 1. It is possible, by a somewhat more careful argument, to lower the bound in the preceding theorem to $m \geq 59$.

2. The statements and proofs of the above results and of the example can be modified so as to apply to Z_3 actions on spheres, since the Whitehead group of Z_3 is zero. In particular, if M^m is an exotic lens space arising from such an action, $N(M^m) < m^2/8 + 1$. On the other hand, it follows from the main lemma in [10] that, for a standard simple lens space L_p^m , with $p \neq 2$, $N(L_p^m) = m^2/4 + m/2 + 1/4$.

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UNIVERSITY OF MASSACHUSETTS, AMHERST, MASSACHUSETTS 01002